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Dynamic theory of relativistic electrons stochastic heating by whistler mode waves with application to the Earth magnetosphere

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[1] In the Hamiltonian approach, electron motion in a coherent packet of the whistler mode waves propagating along the direction of an ambient magnetic field is studied. The physical processes by which these particles are accelerated to high energy are established. Equations governing particle motion are transformed into a closed pair of nonlinear difference equations. The solutions of these equations show there exists a threshold in initial electron energy, below which electron motion is regular and above which electron motion is stochastic. Particle energy spectra and pitch angle electron scattering are described by the Fokker-Planck-Kolmogorov equations. A calculation of the stochastic diffusion of electrons due to a spectrum of whistler modes is presented. The parametric dependence of the diffusion coefficients on the plasma particle density, magnitude of wave field, and the strength of magnetic field is studied. It is shown that significant pitch angle diffusion occurs for the Earth radiation belt electrons with energies from a few keV up to a few MeV.

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1. Introduction

[2] It is suggested that whistler mode waves are responsible for electron acceleration caused by the interaction of these waves and Earth's radiation belt electrons. As a result, electrons may be accelerated up to relativistic energies [*Baker et al.*, 1986]. Thus *Horne and Thorne* [1998] have identified potential whistler wave modes that are capable of resonating with electrons over the important energy range from 100 keV to a few MeV in different regions of Earth's magnetosphere. The statistical aspects of the problem in the quasi-linear approach were discussed by *Summers et al.* [2004]. The basic concept of energy diffusion of relativistic electrons resulting from resonant interaction with whistlers in the magnetosphere has been discussed, for example, by *Walker* [1993], *Reeves* [1996], and *Summers et al.* [1998]. Using the Chirikov overlap criterion [*Chirikov*, 1979; *Karimabadi et al.*, 1990] considered the conditions at which all resonance states break up. They showed that the stochasticity occurs when the amplitude of the obliquely propagating wave is large enough. Electron precipitation caused by chaotic motion due to coupling the bounce motion of nonrelativistic electrons with a large-amplitude whistler wave has been studied by *Faith et al.* [1997].

[3] The goal of this work is to describe the high-energy electron motion in a coherent packet of whistler modes. Stochastic dynamics of charged particles in the field of a wave packet is one of the fundamental problem in the theory of plasma physics [*Lichtenberg and Lieberman*, 1983; *Zaslavsky*, 1998]. Chaotic dynamics of relativistic particles in the spectrum of waves is of particular interest. Thus the stochastic dynamics of relativistic electrons in the time-like wave packet has been discussed by *Chernikov et al.* [1989]. *Krotov and Tel'nikhin* [1998] have shown that the stochastic heating of relativistic particles by the Langmuir waves in space plasmas can be regarded as a possible mechanism for the formation of the energy spectrum of cosmic rays. They also studied the evolution of the distribution function caused by the stochasticity. *Nagornykh and Tel'nikhin* [2002] developed the relativistic theory for the stochastic motion of electrons in the presence of obliquely propagating electrostatic waves. In that case, the ambient magnetic field plays an important role in randomizing the phase of particles with respect to the wave phase. The nonlinear interaction of high-energy electrons with a non-ordinary electromagnetic wave propagating across an ambient magnetic field was investigated in the work of *Zaslavsky et al.* [1991]. In that work a new mechanism of particle diffusion through the so-called stochastic web was described. Stochastic motion of relativistic electrons in the whistler wave packet with application of the results to electron heating in the Jovian magnetosphere was studied by *Khazanov et al.* [2007]. They showed that the Jupiter's radiation belts can be considered as a gigantic natural strange attractor. This attractor has a resemblance with the well-known Lorentz and climatic attractors [*Prigogine*, 1980]; however, it pertains to the nonequilibrium Hamiltonian system. The general mathematical problems of dynamic system with chaotic motion, deterministic chaos and fractal structures have been considered, for example, in the work of

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Arnold and Avez [1968], *Mandelbrot* [1982], and *Schuster* [1984].

[4] The purpose of the present paper is to investigate the dynamics of energetic electrons that are confined by the Earth magnetic field and undergo bounce motion about the equatorial plane. We propose mapping equations which describe the particle motion transiting through a wave packet in a periodic system iteratively. This wave-particle system exhibits chaotic motion when the particle energy exceeds a certain threshold value at given magnitude of the wave field.

[5] The paper is organized as follows. In section 2 we derive the canonical equation of motion in terms of the action-angle variables and discuss physical processes underlying the results. In particular, it is shown that the nonlinear nature of resonance wave-particle interaction leads to two essentially different representation of the whistler wave field (the so-called space-like and time-like wave packet representations). In section 3, a closed set of nonlinear difference equations describing the stochastic motion of relativistic electrons are derived, and dynamics on the strange attractor are studied both analytically and numerically. In section 4 we derive equations that describe the dynamics of resonant nonrelativistic electrons. Solutions of these equations reveal the existence of an energetic threshold below which the particle cannot gain net energy from the wave field. When the initial velocity is above the threshold, the electron moves stochastically and eventually gains a net energy. In section 5 we derive the Fokker-Planck-Kolmogorov equations to describe diffusion in energy, and the effects of radial drift and pitch angle scattering associated with particle stochastic heating. Application of our results to high-energy electrons observed in Earth's radiation belts is described in section 6. In section 7 we give the conclusions of our studies.

2. Basic Equations

[6] Let us consider a relativistic particle of charge $|e|$ and mass m in the wave packet of extraordinary electromagnetic waves propagating along an external uniform magnetic field of strength B. The Hamiltonian corresponding to the problem is

$$H(\mathbf{r},\mathbf{p};t) = \sqrt{m^2 + (\mathbf{p}+\mathbf{A})^2}, \tag{1}$$

and the canonical equations of motion are

$$\dot{\mathbf{p}} = [\mathbf{p}, H], \quad \dot{\mathbf{r}} = [\mathbf{r}, H], \tag{2}$$

where $\mathbf{p}$ is the particle momentum, $\mathbf{r}$ is the position vector, $\mathbf{A} = \mathbf{A}^w + \mathbf{A}^{ext}$ is the vector potential, the superscripts w and ext denote both the wave and external fields, and [,] stand for the Poisson brackets. We have employed here and throughout this paper the system of units in which the speed of light $c = 1$ and charge $|e| = 1$. The Hamiltonian flow (2) acts on a smooth six-manifold $M = R^3 \times R^3$, $\mathbf{p} \in R^3$, $\mathbf{r} \in R^3$, R is the space over the set of all real numbers. By means of the Maxwell equations

$$\mathbf{B} = rot\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t, \tag{3}$$

we get the following relations:

$$\mathbf{r} = (x, y, z), \quad \mathbf{B}^{ext} = (0, 0, B), \tag{4}$$

$$\mathbf{A}^w = \left(\sum_k A_k \sin(zk - t\omega_k), \ \sum_k A_k \cos(zk - t\omega_k), 0\right), \tag{5}$$

$$\mathbf{A}^{ext} = (-By, Bx, 0)/2, \tag{6}$$

written in a Cartesian spatial coordinates system whose z axis is directed along the external magnetic field. Here the expression for $\mathbf{A}^{ext}$ is written in the axial gauge, A_k is the amplitude of a mode in the wave packet, k is the wave number, and ω_k is the dispersion equation.

[7] The dispersion relation for the electron branch of whistler mode waves in the cold magnetoplasma is written as

$$k^2/\omega^2 = 1 + \omega_p^2/[\omega(\omega_B - \omega)], \tag{7}$$

where ω_B and ω_p are the gyrofrequency and electron plasma frequency, respectively. This equation in a long-wavelength approximation $(\omega_B\omega/\omega_p^2) \ll 1$ reduces to

$$v_{ph}^2 = \omega(\omega_B - \omega)/\omega_p^2, \quad v_{ph}^2 \ll 1. \tag{8}$$

We apply equations (4)–(6) to (1) and carry out a canonical transformation $(x, p_x; y, p_y) \rightarrow (\theta, I)$,

$$x = r\cos\theta, \qquad p_x = -(mr\omega_B/2)\sin\theta, \tag{9}$$

$$y = r\sin\theta, \qquad p_y = (mr\omega_B/2)\cos\theta; \tag{10}$$

$$r = \sqrt{2m\omega_B I}/m\omega_B, \qquad \omega_B = B/m, \tag{11}$$

where r is the gyroradius. The Hamiltonian of system in terms of the new variables, an action (I), and an angle (θ), is given by

$$H(z, p_z; \theta, I; t) = H_0(p, I) + \sqrt{2m\omega_B I}H_0^{-1} \cdot \sum_k A_k \cos(zk + \theta - t\omega_k), \tag{12}$$

$$H_0(p, I) = \sqrt{m^2 + p^2 + 2m\omega_B I}, \quad p \equiv p_z. \tag{13}$$

Here we have assumed that the ratio $\mu = A/m$ ($\ll 1$) is the small parameter of the problem and retain in (12) only the leading terms.

[8] The equations of motion associated with (12) are

$$\dot{p} = [p, H] = \sqrt{2m\omega_B I}H_0^{-1} \cdot \sum_k kA_k \sin\psi, \tag{14}$$

$$\dot{I} = [I, H] = \sqrt{2m\omega_B I}H_0^{-1} \cdot \sum_k A_k \sin\psi, \tag{15}$$

$$\dot{z} = [z, H] = pH_0^{-1}, \quad \dot{\theta} = [\theta, H] = \omega_B m H_0^{-1}. \tag{16}$$

In (16) we omit the terms of the order of μ^2 and introduce the definition for the phase

$$\psi \overset{def}{=} zk + \theta - \omega_k t. \tag{17}$$

Now the structure of wave packet, $A^w(t, z) = \Sigma_k A_k \exp[i(zk + \theta - t\omega_k)]$, is to be specialized. We will discuss first dynamics of a relativistic electron whose velocity along the direction of an ambient magnetic field, v_z, is close to the speed of light. Then by virtue of dispersion relation, the inequality $(v_{ph}/v_z)^2 \ll 1$ is valid, therefore the shape of wave packet can be given in the so-called space-like (SL) representation [*Zaslavsky*, 1998; *Khazanov et al.*, 2007],

$$A^w(t,z) = A\exp(i\psi)\sum_{n\in Z}\delta(\zeta/L - n), \tag{18}$$

where the Poisson sum formula

$$\sum_{n\in Z}\exp(in\Delta kz) = \sum_{n\in Z}\delta(\zeta/L - n) \tag{19}$$

$$\psi = zk + \theta - \omega t. \tag{20}$$

has been employed. Here A, ω, and k are the magnitude, frequency, and wave number of the fundamental (characteristic) mode, $\zeta = (z/L)(\text{mod } 1)$, L is the characteristic spacescale, $\delta_n \equiv \delta(\zeta - n)$, $\delta(\cdot)$ is the Dirac delta function, and Z denotes the set of all integers. Note that such a wave packet manifests itself as a periodic sequence of impulses with characteristic spatial period $L = 2\pi/\Delta k$, where $\Delta k/k = 2\pi/N$, $N(\gg 1)$ is a characteristic number of modes in the wave packet.

[9] In the limit, $\Delta k \to 0$, a wave packet can be approximated by the time-like (TL) representation

$$A^w(t,z) = A\exp(i\psi)\sum_{n\in Z}\delta(t/T - n), \tag{21}$$

where T is the timescale of the problem. Note that both representations are often used and that the TLR is available for the problem if the condition $\eta^2 \ll 1$ holds. The TLR of the electric field of the electrostatic waves was used by *Chernikov et al.* [1989] to derive the relativistic generalization of the standard map. On the other hand, *Klimov and Tel'nikhin* [1995] use SLR to describe the stochastic motion of relativistic particle in the electrostatic field of Langmuir waves.

3. Particle Dynamics in a Space-Like Wave Packet

[10] We specify first the wave spectrum of a packet. Let us assume that the wave packet is given by (18). Thus, we will consider relativistic electron motion in the space-like packet (SLP) of the whistler mode waves. In this approach, we write down the equations of motion (14)–(16) in the form:

$$\dot{p} = kA\sqrt{2m\omega_B I}H_0^{-1}\sin\psi\sum_{n\in Z}\delta(\zeta - n), \tag{22}$$

$$g^t:\quad \dot{I} = \sqrt{2m\omega_B I}H_0^{-1}A\sin\psi\sum_{n\in Z}\delta(\zeta - n), \tag{23}$$

$$\dot{z} = pH_0^{-1},\quad \dot{\theta} = \omega_B m H_0^{-1}; \tag{24}$$

$$H_0(p,I) = \sqrt{m^2 + p^2 + 2m\omega_B I}, \tag{25}$$

$$\dot{\psi} = \omega(p,I) = kpH_0^{-1} + \omega_B m H_0^{-1} - \omega. \tag{26}$$

Here p, I, z and θ are the coordinates on $M^4 = R^2 \times T$, $p \in R$, $I \in R$, and (z (mod L), θ (mod 2π)) $\in T = S \times S$.

[11] First we observe that the phase flow g^t on M^4 given by equations (22)–(24) is invariable under the translation of a phase point with respect to z (mod L), and possesses the integral invariant of motion

$$p - kI = inv. \tag{27}$$

This allows us to represent evolution of the phase flow as a certain iterative process, $g^{t+T_n} = g^t g^{T_n}$, or rather, as a successive action of the one-parametric group of transformations g^{nT_n} on a two-dimension manifold M^2, where T_n is a time step, which will be determined later. Let the variables z and p represent a coordinate pair on the reduced space of orbits,

$$2\omega_B I = \alpha p,\quad \alpha = (2\omega_B/\omega)v_{ph}. \tag{28}$$

We set the constant of integration equals zero, because the phase flow, being an analytical invariant, is invariable under the transformation $H \to H + const$.

[12] Now the explicit form of iteration system is to be found. Previously, equations (22)–(26) have been analyzed by the authors in the work of *Khazanov et al.* [2007] through the technique of calculation on a fibre bundles. In the present paper we will formulate the problem in terms of differential 1-form. Denote by $\tilde{d}p$, $\tilde{d}\zeta$, $\tilde{d}t$ the coordinate basis of one-form on an extended phase space [*Schutz*, 1982], and introduce the new variables,

$$\varepsilon_z = p/m,\ \varepsilon_t = \sqrt{2m\omega_B I}/m,\ \varepsilon = E/m, \tag{29}$$

where $E = \sqrt{m^2 + p^2 + 2m\omega_B I}$ is the particle energy. Then, using the further result

$$L\tilde{d}\zeta = pH_0^{-1}\tilde{d}t, \tag{30}$$

we represent equations (22) and (26) as

$$\tilde{d}\varepsilon_z - \varepsilon_z^{-1}\varepsilon_t N(A/m)\sin\psi\sum_{z\in Z}\delta_n\tilde{d}\zeta = 0, \tag{31}$$

$$\tilde{d}\psi - \left(\omega - \omega_B\varepsilon^{-1} - k\varepsilon_z\varepsilon^{-1}\right)L(\varepsilon/\varepsilon_z)\tilde{d}\zeta = 0. \tag{32}$$

The quantity A/m has a clear physical meaning: A/m is the dimensionless representation of the ratio of the work of the wave field at one wavelength to the particle rest energy.

Regarding the relationship (3) between fields A^w and B^w, it is convenient to represent A/m in the form

$$A/m = \alpha b/2, \quad b = B^w/B. \tag{33}$$

Making use of the invariant of motion (28), we integrate one by one resulting equations to obtain the closed set of nonlinear difference equations

$$\begin{aligned} v_{n+1} &= v_n + \alpha^{3/2} N b \sin\psi_n, \\ \psi_{n+1} &= \psi_n + N\Big(1 + 1/2\cdot\alpha|v_{n+1}^{-2/3}| \\ &\quad -|v_{n+1}^{-2/3}|v_{ph}\sqrt{1 + v_{n+1}^{4/3} + \alpha v_{n+1}^{2/3}}\Big) sgn\, v_{n+1} (\mathrm{mod}\ 2\pi), \end{aligned} \tag{34}$$

expressed in terms of the variables v, ψ, where v_{n+1} and v_n are the values of the normed momentum at times $(n + 1)$ and n, respectively, and $\psi_{n+1} - \psi_n$ is the phase shift acquired by the particle. Then integrating (30) we obtain the time step, $T(n)$,

$$t_{n+1} - t_n = T(n), \quad T(n) = LE_{n+1}/|p_{n+1}|, \tag{35}$$

which is a function of n. We have employed here the following notations:

$$v = |\varepsilon_z|^{3/2} sgn\, \varepsilon_z, \quad N = [kL]. \tag{36}$$

Here N is the characteristic number of modes in the SL packet.

[13] Equation (34) admit solutions describing stochastic behavior. We will show this using the overlap criterion [*Zaslavsky*, 1998]

$$\left|\frac{d\psi_{n+1}}{d\psi_n} - 1\right| \geq 1. \tag{37}$$

Applying (37) to (34) yields the allowed domain of

$$\varepsilon_z \leq \varepsilon_b, \quad \varepsilon_b = \alpha\left(\frac{1 - v_{ph}}{3} bN^2\right)^{2/5}, \tag{38}$$

in which the criterion is satisfied.

[14] This test actually predicts chaotic motion of the system within a certain domain of phase space. The extent of the domain tends to zero as $d\varepsilon_b/db \to b^{-3/5}$, as $b \to 0$.

[15] Let $\varepsilon_b^2 \gg 1$, by which we stipulate for the ultra-relativistic limit. This allows us to neglect the term $v_{ph}m^2/2p^2$ in the second equation of (34). Now $T(n)$ tends to the limit, $T = L$, and the set of equations (34) goes over into the map:

$$g^n: \quad \begin{aligned} u_{n+1} &= u_n + Q\sin\psi_n, \\ \psi_{n+1} &= \psi_n + 3/2|u_{n+1}|^{2/3} sgn\, u_{n+1} \quad (\mathrm{mod}\ 2\pi), \end{aligned} \tag{39}$$

written in the perception

$$u = \left(\frac{3\varepsilon_z}{(1 - v_{ph}\alpha N)}\right)^{3/2}, \ Q = \left(\frac{3}{1 - v_{ph}}\right)^{3/2} \frac{b}{N^{1/2}}. \tag{40}$$

Note that the map g^n is invariable under the inversion of a point with respect to a circle ψ (mod $2\pi \in S$) and the transformation

$$u \to -u, \quad \psi \to -\psi. \tag{41}$$

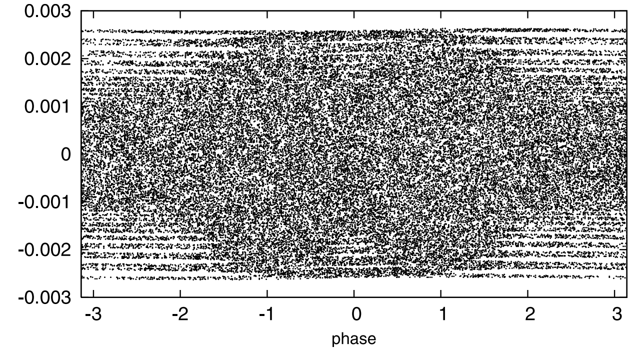

Figure 1a. Strange attractor of the system at $Q = 5 \cdot 10^{-5}$.

These equations are formally equivalent to those obtained by the authors in the work of *Khazanov et al.* [2007]. However, in the present case the map g^n takes account of the small but finite effects of wave dispersion. This effect is included particularly in the parameter Q. When we take the limit $v_{ph} \to 0$ the result of Khazanov et al. will follow with proper changes of units.

[16] The dynamical system (M, g^n) has been studied by the authors in the work of *Khazanov et al.* [2007]. In this work was shown that evolution of the system is realized on an strange attractor (SA). Shown in Figure 1 is the strange attractor of the system. According to *Khazanov et al.* [2007], the strange attractor is the invariant set, $g^nSA = SA$, $n \to \infty$, embedded in the phase space.

[17] Now we turn to the Jacobi matrix, J, of g^n

$$J = \frac{\partial(u_{n+1}, \psi_{n+1})}{\partial(u_n, \psi_n)} \tag{42}$$

to understand how the group g^n acts on the smooth manifold M. As we discussed above, we are considering the dynamics as a successive action of g^n on M. It is important to note that the Jacobian of (42) is equal to one; therefore, g^n has a structure of the differentiable area-preserving map, expressed in terms of variables u and ψ, being the canonical pair.

[18] The relation $|tr\ J| = 3$, $tr\ J$ is the trace of the matrix, is known to be the test that corresponds to topological modification of a phase space [*Arnold and Avez*, 1968], which, in our case, well defines the upper bound of $\{u\}$. So far as,

$$\mathrm{tr}J = 2 + Qu^{-5/3}, \tag{43}$$

therefore

$$\sup\{u\} = u_b, \quad |\,u_b\,| = Q^{3/5}. \tag{44}$$

This agrees with the numerical solution represented in Figure 1. The strange attractor is characterized by two invariants, the Kolmogorov entropy, being the kneading invariant, and the fractal measure, d_f, the values of which ensue from the relations, $det\ J = 1$ and $tr\ J = 3$. The rate of intermixing, or in other words, the mean rate of loss of information is given by the Kolmogorov entropy, $K = \ln((3 + \sqrt{5})/2))$, and structural stability is determined by the

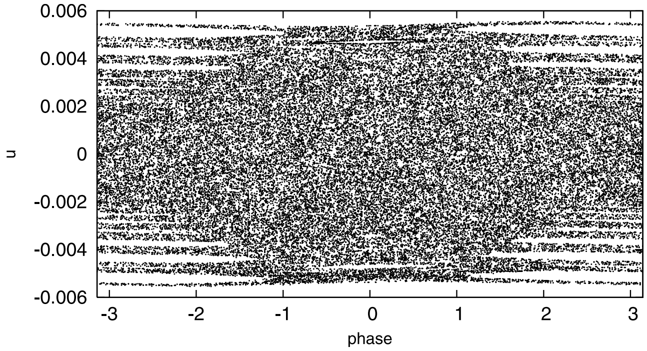

Figure 1b. Strange attractor of the system at $Q = 5\pi \cdot 10^{-5}$.

fractal measure, $d_f = 2$, which implies that the phase space is dense almost everywhere, and points of the phase curve evenly fill all obtainable phase space (strange attractor).

[19] Considering (40), from (44) the following equation results:

$$\varepsilon_b = \alpha \left(\frac{1 - v_{ph}}{3} bN^2 \right)^{2/5}, \tag{45}$$

which well determines the upper value of the energy spectrum. This is also in good agreement with the results of qualitative analysis.

4. Nonrelativistic Electron Motion in a Time-Like Wave Packet

[20] We now discuss the dynamics of nonrelativistic electrons in the wave packet of the whistler mode. In this case, the condition $v_z/v_{gr} < 1$ is valid, therefore by means of (21) the wave field of a packet may be approximated as

$$A^w(z,t) = A\cos(kz - \omega_B t - \omega t) \sum_n \delta(\tau - n). \tag{46}$$

Here $\tau = t/T$, T is the timescale of the problem.

[21] Substituting this expression in (11), we write down the Hamiltonian as

$$H(z,p;\,\theta,I;\,t) = H_0(p,I) + \sqrt{2\omega_B I/m}A\cos\psi \cdot \sum_{n\in Z} \delta(\tau - n), \tag{47}$$

$$H_0 = p^2/(2m) + \omega_B I, \quad mv_t^2/2 = \omega_B I, \tag{48}$$

$$\psi = zk + \theta - t\omega. \tag{49}$$

The equations of motion associated with (47) are

$$\dot{p} = k\sqrt{2\omega_B I/m}A\sin\psi \sum \delta(\tau - n), \tag{50}$$

$$\dot{I} = \sqrt{2\omega_B I/m}A\sin\psi \sum \delta(\tau - n) \tag{51}$$

$$\dot{z} = v_z = p/m, \quad \dot{\theta} = \omega_B. \tag{52}$$

In (52) we retain only the leading terms on the condition that A/mv_t is a small parameter of the problem.

[22] Since the whistler wave frequency ω is smaller than the electron cyclotron frequency ω_B, electrons in general must move in the opposite direction to the waves in order to match the resonance condition

$$kv_z + \omega_B - \omega = 0. \tag{53}$$

In view of this fact and also taking into account the relation $\omega_B I = mv_t^2/2$, we write the invariant of Hamiltonian flow (50–52) as

$$v_t^2 = \alpha(v_r - |v_z|), \quad \alpha = 2(\omega_B/\omega)v_{ph} \tag{54}$$

where $v_r = v_{ph}(\omega_B - \omega)/\omega$ is the electron resonance speed.

[23] To understand the physical picture of the stochastic motion, we use the overlap criterion [*Chirikov*, 1979]. In the broad wave spectrum case given by (19), the condition (53) describes the family of the resonance states. First, varying this equation, we evaluate the interval between adjacent resonance states in the velocity space as $k\Delta v_z \simeq \Delta\omega = T^{-1}$, and

$$\Delta v_z \simeq v_{ph}/N, \quad N = [\omega T], \tag{55}$$

where N is the characteristic number of modes in the TL packet. One can then use equation (50) to estimate the width, δv_z, of a resonance state in the velocity space,

$$\delta v_z \simeq \alpha bNv_t/2v_{ph}, \tag{56}$$

where we have again used the notation $A/m = \alpha\, b/2$, $b = B^w/B$. Now from the overlap criterion, $\delta v_z \geq \Delta v_z$, and the two above expressions the following relation results

$$v_t \geq v_c = v_{ph}\omega/\omega_B N^2 b. \tag{57}$$

Note the term in the RHS of equation (50) corresponds to the Lorentz force experienced by a spiralling electron in a wave magnetic field. Consequently, we conclude from the above equation that stochasticity occurs when the Lorentz force acting in the axial direction on an electron exceeds some defined value.

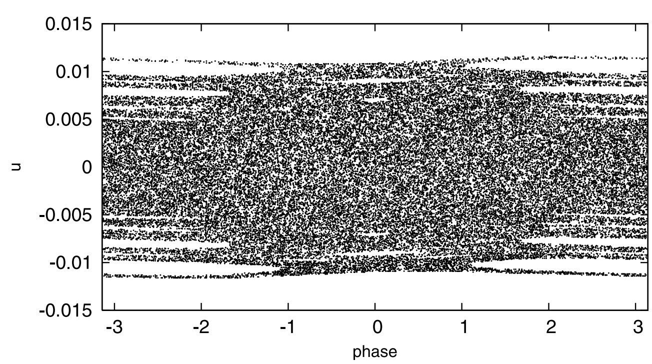

Figure 1c. Strange attractor of the system at $Q = 5\pi^2 \cdot 10^{-5}$.

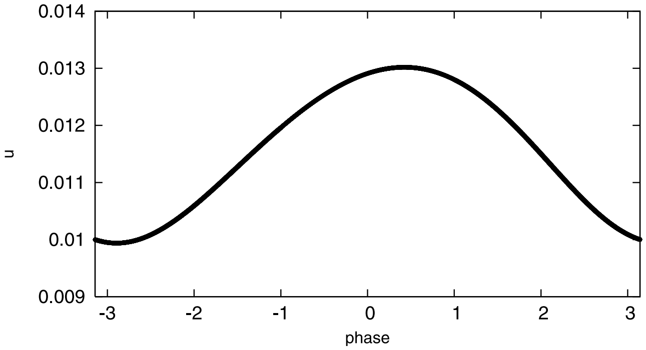

Figure 2a. Phase space of map G^n. A single trajectory of length 10^6 for $Q = 2 \cdot 10^{-2}$, $s = 2.5 \cdot 10^2$, $(u_0, \psi_0) = (0.01, 0.0001)$.

[24] Now we may be able to estimate the value of the wave field at which electron motion becomes stochastic. We substitute the value of v_c from (57) in (54) at $v_z = 0$, to obtain

$$b \geq b_c = \left(v_{ph}/2v_r\right)^{1/2}(\omega/\omega_B)^{3/2}N^{-2}. \tag{58}$$

[25] Now we turn to the equations of motion. We use (54) and the expression $mv_t^2/2 = \omega_B I$ once more to represent the equations in the form

$$\dot{v}_t = \frac{1}{2}\omega_B \alpha b \sin\psi \sum_{n \in Z} \delta(\tau - n), \tag{59}$$

$$\dot{\psi} = \omega(v_t, \omega) = \omega_B - \omega - kv_r + (k/\alpha)v_t^2. \tag{60}$$

[26] By integrating these equations can be transformed into a map, G^n,

$$G^n : \begin{array}{l} u_{n+1} = u_n + Q\sin\psi_n, \quad \{u \in U \subset R| \;\; 0 < u \leq 1\}, \\ \psi_{n+1} = \psi_n + su_{n+1}^2 \pmod{2\pi}, \end{array} \tag{61}$$

where subscript n refers to values taken at time $t = nT$, and the new variable, u, is introduced by the relation

$$u = v_t/v_0, \quad v_0 = \sqrt{\alpha v_r}. \tag{62}$$

Here v_0 is the perpendicular (to B) speed of electron, and the new parameters are

$$Q = \frac{1}{2}N\alpha b\omega_B/v_0\omega, \quad N = [\omega T], \quad s = Nv_0^2/\alpha v_{ph}. \tag{63}$$

We have numerically integrated equations (61) for values of Q from 0.02 to 0.2. Our results are shown in Figure 2.

[27] The overall picture of the phase space is quite different for $u < u_c$ and $u \geq u_c$. In the first case, the motion is regular. These figures indicate the existence of a threshold for the initial particle velocity above which the trajectory becomes chaotic and the extent of the stochastic region increases linearly with Q.

[28] Thus the Jacobian of the matrix of Q^n, $\det J = 1$, therefore G^n is the measure-preserving map, and ψ, u are the symplectic pair on the smooth manifold, M: $U \times S$. Thereby we apply the condition $tr\ J = 3$ to (61), to find the expression

$$u_c = inf\{u\} = (2Qs)^{-1}, \tag{64}$$

that well determines the lower bound $inf\{u\}$ of the stochastic set.

[29] Substituting (62) and (63) in (64), we get

$$v_c = v_{ph}\omega/\omega_B N^2 b, \tag{65}$$

which is in good agreement with the numerical solutions and results of qualitative analysis. So we resume that nonlinear electron acceleration by a wave packet of the whistler mode waves is always a stochastic process.

5. Diffusion Evolution

[30] We recognize that the dynamics of the system exhibits a random walk, and all points of the phase trajectory tend to a certain strange attractor (SA) at $n \to \infty$. In this case, the nature of evolution to the steady state is so-called deterministic diffusion [*Lichtenberg and Lieberman*, 1983]. As the canonical status of the phase variables has been demonstrated above, therefore the distribution function (probability density)$f(u;\, t)$ obeys the Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\partial f(u;t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial u}D\frac{\partial f}{\partial u}. \tag{66}$$

Here D is the conventional diffusion coefficient in phase space,

$$D = \left\langle (u_{n+1} - u_n)^2 \right\rangle T^{-1}, \tag{67}$$

in which $(u_{n+1} - u_n)$ is substituted from either g^n or G^n, and $\langle\cdot\rangle$ denotes the phase average, T is the timescale of the problem. $f(u, t)$ is a differentiable function supported in $\{U\}$ with the norming

$$\int_{u \in \{U\}} f(u,t)du = 1, \tag{68}$$

$\{U\}$ is a range of the variable u.

[31] First, by means of g^n (or G^n) we calculate the diffusion coefficient

$$D = Q^2/2T, \tag{69}$$

and evaluate the characteristic time for redistribution of u over the spectrum

$$t_d \simeq 2u_m^2/D = T(2u_m/Q)^2. \tag{70}$$

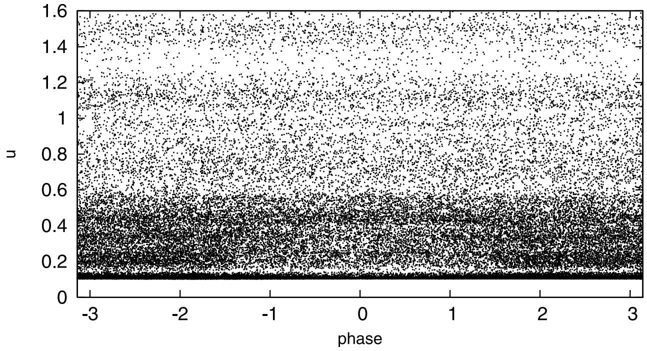

Figure 2b. Phase space of map G^n. A phase curve for the same values of Q and s as in (a). The trajectory was started from the point (0.11, 0.0001).

In order that the FPK equation is valid, the amount of changes in u, $\Delta u \simeq Q$, must be smaller than the upper value of the u-spectrum, u_m, thus the diffusion coefficient D is local in phase space. In other words, D fits a FPK equation when t_d is much larger than T. This obviously imposes limitations on permissible values of Q.

[32] We will determine the evolution of the system for $t \leq t_d$. We exploit the FPK equation with $f(u)$ and its derivative $\partial f/\partial u$ vanishing at the upper and lower boundaries. We introduce the moment $\langle u^2 \rangle = \int_{-\pi}^{\pi} du u^2 f(u)$, multiply equation (66) by u^2, and integrate the resulting equation over u to obtain

$$d\langle u^2 \rangle / dt = D, \quad u \in \{U | u \leq u_m\}. \tag{71}$$

The restriction on u is needed because the spectrum is bounded above.

[33] The distribution function $f(u, t)$ in the non-relativistic case obeys the FPK equation thus the coordinates (u, ψ) are proper variables of (54). By reason of (64), we write the normalization as

$$\int_{u_c}^{1} f(u)\, du = 1. \tag{72}$$

Then by means of (63) along with (64), we calculate by formulas (67) and (70) the diffusion coefficient

$$D = Q^2/2T = \alpha^2 N^2 b^2 \omega_B^2 / 8T v_0^2 \omega^2, \tag{73}$$

and the characteristic time for redistribution of u over the spectrum

$$t_d = 4T/Q^2 = 16T v_0^2 \omega^2 / N^2 \alpha^2 b^2 \omega_B^2. \tag{74}$$

Next using the normalization for $f(u, t)$ in the FPK equation, it follows the distribution function for $t > t_d$ can be given by

$$f(u)\, du = (1 - u_c)^{-1}\, du. \tag{75}$$

This means that the random variable u is evenly distributed on $[u_c, 1]$.

[34] As we shall show later under typical conditions, $u_c \ll 1$. Considering this fact, from (75), the transformation $f(u)du = f(v)\, dv$ and the relation $u = v_t/v_0$ we obtain

$$f(v_t) = v_0^{-1}. \tag{76}$$

Taking account of (72), by (76) we derive the quadratic mean of v_t

$$\begin{aligned} \langle v_t^2 \rangle &= \int_0^{v_0} f(v_t) v_t^2\, dv_t = v_0^2/3 \\ &= 4/3 \cdot v_{ph}^2 (\omega_B - \omega)/\omega, \end{aligned} \tag{77}$$

the mean value of $|v_z|$ and the mean velocity

$$\langle |v_z| \rangle = 2/3 \cdot v_{ph} (\omega_B - \omega)/\omega_B, \tag{78}$$

$$\langle v_z \rangle = 0. \tag{79}$$

Then we define by

$$T_t = \langle m v_t^2 / 2 \rangle, \quad T_z = \langle m v_z^2 / 2 \rangle, \tag{80}$$

to find the ratio

$$T_t/T_z = 5\alpha/8v_r = 5/8 \cdot \omega_B^2 / \omega(\omega_B - \omega), \tag{81}$$

describing the anisotropy of distribution (54) in the (v_z, v_t) phase space. Anisotropic distribution function (54) with its derivative

$$\frac{dv_t}{dv_z} = -\frac{1}{2}\sqrt{\alpha/(v_r - |v_z|)}\, sgn\, v_z, \tag{82}$$

that tends to infinity as $|v_z| \to v_r$, describe the so-called pancake distribution in the (v_z, v_t) phase space.

[35] It is useful to evaluate the energy distribution function, $f(\varepsilon)$, in the range of nonrelativistic energies. Considering (54), we note that the particle energy E is a specified function of v_t. Thus the measure-preserving transformation

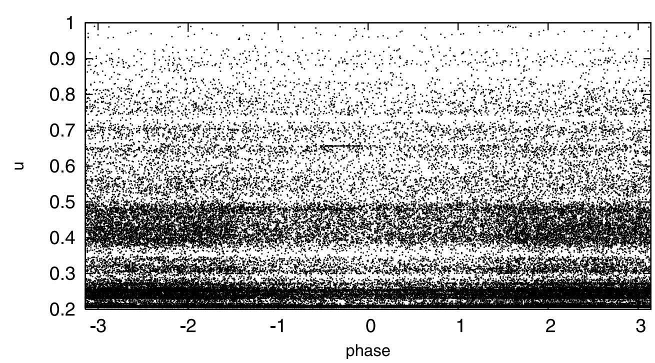

Figure 2c. Phase space of map G^n. One trajectory of length $7 \cdot 10^6$ for $Q = 10^{-2}$, $s = 2.5 \cdot 10^2$, $(u_0, \psi_0) = (0.21, 0.0001)$.

$f(\varepsilon)\ d\varepsilon = f(v_t)\ dv_t$ determines this problem completely, subject to appropriate boundary conditions. To that end, we have to find the lower (ε_c) and upper (ε_0) boundaries of the energetic spectrum. At first, from (54) we evaluate the threshold value of particle speed, v_{th},

$$v_{th} = |v_z|_c = v_r - v_c^2/\alpha. \tag{83}$$

Now it is clear that ε_c and ε_0 are given by

$$\varepsilon_c = 1/2 \cdot \left[\left(v_r - v_c^2/\alpha\right)^2 + v_c^2\right], \tag{84}$$

$$\varepsilon_0 = v_0^2/2 = \alpha v_r/2. \tag{85}$$

The threshold energy ε_c is a function of magnitude of wave field, b, and, in view of (64) ε_c hinges on b as follows:

$$\varepsilon_c \to \infty \ \text{ as } \ b^{-4}, \ \text{ as } \ b \to 0, \tag{86}$$

$$\left(\varepsilon_c - v_r^2/2\right) \to 0 \ \text{ as } \ b^{-2}, \ \text{ as } \ b \to \infty. \tag{87}$$

The corresponding solution for $f(\varepsilon)$ provided that

$$\int_{\varepsilon_c}^{\varepsilon_0} f(\varepsilon)\, d\varepsilon = 1$$

can be expressed in the form

$$f(\varepsilon)\, d\varepsilon = 1/2 \cdot [(\varepsilon_0 - \varepsilon_c)(\varepsilon - \varepsilon_c)]^{-1/2}\, d\varepsilon, \ \varepsilon_0 > \varepsilon > \varepsilon_c. \tag{88}$$

This solution relates directly to the behavior of the system near the order-chaos bifurcation transition. The function $f(\varepsilon)$ undergoes a sudden change at $\varepsilon = \varepsilon_c$, and in the region of regular motion ($\varepsilon < \varepsilon_c$, ε = const) has the form of the Dirac δ- function, $f(\varepsilon) = \delta\ (\varepsilon - \text{const})$.

[36] Thus $f(\varepsilon)$ describes the density of states in the energy space, the mean particle energy and relative standard deviation can be calculated by

$$\int_{\varepsilon_c}^{\varepsilon_0} \varepsilon f(\varepsilon)\, d\varepsilon = 1/3(\varepsilon_0 - \varepsilon_c) + \varepsilon_c, \tag{89}$$

$$\frac{\sqrt{\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2}}{\sqrt{\langle\varepsilon^2\rangle}} = 0.4. \tag{90}$$

The latter indicates the high level of fluctuations in the energetic spectrum. Evolution of all means for $t < t_d$ is governed by (62), where t_d is given by (74).

[37] Equation (71) in the explicit form may be written as

$$\frac{d}{d\,t}\langle v_t^2\rangle = \alpha^2 b^2 N^2 \omega_B^2/8T\omega^2. \tag{91}$$

Then we have respect to the invariant of motion along with (65), and define by $\mu_a = v_t^2/2B$ the first adiabatic invariant to find the rate of a change of μ_a

$$\dot{\mu}_a = \alpha^2 b^2 N^2 \omega_B^2/16TB\omega^2, \tag{92}$$

and the heating rate

$$\dot{\varepsilon} = D_\varepsilon, D_\varepsilon = \alpha^2 b^2 N^2 \omega_B^2/16T\omega^2. \tag{93}$$

This result is nontrivial, because chaotic motion in the (u, ψ) phase space leads to important and easily observable macroscopic effects such as the stochastic heating of plasma particles.

[38] Now we describe the effects associated with stochastic heating of high-energy particles. First, we discuss the pitch angle distribution in a range of nonrelativistic energies.

[39] Denote by χ_p the pitch angle, and let χ be its complementary angle. Then we define

$$\tan\chi = v_z/v_t, \quad \chi \in (-\pi/2, \pi/2), \tag{94}$$

and use equations (54) to represent tanχ as a function of $u(u = v_t/v_0)$, namely,

$$\tan\chi = (v_r/\alpha)^{1/2} \cdot \left(1 - u^2\right)/u. \tag{95}$$

Then via (75) and the measure-preserving point transformation $f(u)\ du = f(\chi)d\chi$, we find the angle distribution

$$\begin{aligned} f(\chi) = (\alpha/v_r)^{1/2} \Big(& u^2/\left(1+u^2\right) \\ &+ (v_r/\alpha)\left(1-u^2\right)^2/\left(1+u^2\right)\Big). \end{aligned} \tag{96}$$

Equation (96) determines the angle distribution as a function of u. Nonlinear transformation (95) allows us to express (96) as a function of a proper variable χ. Note that $f(\chi)$ is invariable under the transformation

$$u \ \to -u, \quad \chi \ \to -\chi; \ \ f(\chi) = f(-\chi), \tag{97}$$

therefore $f(\chi)$ is a symmetric function on ($-\pi/2$, $\pi/2$). Unfortunately, the explicit expression for $f(\chi)$ has a very complicated form. The following asymptotic formulas are valid:

$$\begin{aligned} f(\chi) &\approx 1/2 \cdot (\alpha/v_r)^{1/2}\left(1 - (\alpha/v_r)^{1/2}\tan\chi\right) \\ &\text{as } \ \chi \to 0, \end{aligned} \tag{98}$$

$$f(\chi) \approx (v_r/\alpha)^{1/2}\left(1 + \tan^{-1}\chi\right), \ \text{ as } \ \chi \to \pm\pi/2, \tag{99}$$

$$\begin{aligned} f(\chi) &\approx (v_r/\alpha)^{1/2}(1 + 2(\varepsilon/\varepsilon_c - 1)) \\ &\text{as } \ \varepsilon \downarrow \varepsilon_c, \ \chi \to \pm\pi/2, \end{aligned} \tag{100}$$

$$\begin{aligned} f(\chi) &\approx 1/2 \cdot (v_r/\alpha)^{-1/2}\left(\sqrt{\varepsilon/\varepsilon_c} - 1\right) \\ &\text{as } \ \varepsilon \uparrow \varepsilon_0, \ \chi \to 0. \end{aligned} \tag{101}$$

where ε_c and ε_0 are given by (84) and (85). Making use of these equations, we conclude that the function $f(\chi)$ governing the distribution of angles is a convex symmetric function of χ, having the maximum at $\chi = 0$ ($\chi_p = \pi/2$). The result indicates also that particles are equally likely to be scattered in the direction with or against the wave. To this may be added that the given function is an increasing function of the particle energy on (ε_c, ε_0), and it hinges asymptotically on ε as ($\varepsilon - \varepsilon_c$) near the energetic threshold, and as $\sqrt{\varepsilon/\varepsilon_c} - 1$ for $\varepsilon > 2\varepsilon_c$.

[40] Since the magnetic field is constant, the gyroradius is a direct measure of the perpendicular electron velocity. Thus the stochastic heating will be accompanied by a radial drift of particles in space. Indeed, in view of the relation $r = v_t/\omega_B$, via (91), we get the following expression:

$$\frac{d\left\langle r^2\right\rangle}{d\,t} = \omega_B^{-2}\frac{d\left\langle v_t^2\right\rangle}{d\,t} = D_t, \tag{102}$$

$$D_t = \alpha^2 b^2 N^2/16T\omega_B^2, \quad \varepsilon > \varepsilon_c, \tag{103}$$

where D_t is the coefficient of collisionless diffusion across the ambient magnetic field. Equations (102) and (103) are correct if the inequality $r/R_E = (\omega v_t/\omega_B\, v_{ph})/kR_E \ll 1$ holds. Thus $\omega\, v_t/\omega_B v_{ph} \sim 1$, and $kR_E \gg 1$, R_E is the Earth's radius, therefore the condition, $r/R_E \ll 1$, is fulfilled trivially. Note that the basic equations are written in the very approximation of locally uniform field, namely $kR_E \gg 1$, $\omega\, T \gg 1$.

[41] We now turn to the case of diffusion in the range of relativistic energies. Considering (70), and inserting (44) in (70), we obtain

$$t_d = 4TQ^{-4/5}, \tag{104}$$

or, in the explicit form

$$t_d = 4T\left(\frac{1-v_{ph}}{3}\right)^{6/5}\left(\frac{N}{b^2}\right)^{2/5}. \tag{105}$$

Now we need to show that the FPK description in this case is also valid. The test $t_d \gg T$ determines the following range of permissible values of Q,

$$Q \ll 1. \tag{106}$$

At this rate, the magnitude of b satisfies

$$b < 4^{5/4}\left(\frac{1-v_{ph}}{3}\right)^{3/2} N^{1/2}. \tag{107}$$

Typically, $b < 1$, $v_{ph}^2 \ll 1$, $N \gg 1$, that is, the test (106) is fulfilled trivially, and consequently a FPK approximation is applicable in a range of relativistic energies.

[42] To describe the process of energization of electrons, we substitute the definitions (40) for u and Q in (71), to find

$$\frac{d}{dt}\left\langle \varepsilon^3\right\rangle = \frac{1}{2}\alpha^3 N^2 b^2. \tag{108}$$

This equation allows us to evaluate the heating rate

$$\frac{d}{dt}\varepsilon \simeq \frac{1}{6}\alpha^3 N^2 b^2 \varepsilon^{-2}. \tag{109}$$

The solutions of equations (108) and (109) indicate that the heating rate decreases with time as $t^{-2/3}$, and is a rapidly decreasing function of ε.

[43] We now turn to the problems of radial drift and scattering of relativistic electrons by whistlers. First with the help of the invariant of motion (28), we write down expression (11)

$$r(\varepsilon) = \sqrt{\alpha\varepsilon}/\omega_B, \quad r \le r_b, \tag{110}$$

$$r_b = \frac{\sqrt{\alpha\varepsilon_b}}{\omega_B} = \frac{\alpha}{\omega_B}\left(\frac{1-v_{ph}}{3}N^2 b\right)^{1/5}, \tag{111}$$

where r_b is the largest possible value of r.

[44] Then it proves more convenient to represent $\tan\chi$ in the form

$$\tan\chi(\varepsilon) = \sqrt{\frac{\varepsilon}{\alpha}}, \quad |\chi| \le \chi_b, \tag{112}$$

$$\tan\chi_b = \left(\frac{1-v_{ph}}{3}N^2 b\right)^{1/5}, \tag{113}$$

which is valid provided that $A/m \ll 1$.

[45] It should be noted from the above two equations that r and χ are related by

$$2\tan\chi = kr. \tag{114}$$

The implications of events points out the role of cooperative effects in the wave-particle system.

[46] As appears from these equations changes in ε lead to changes in r and χ, therefore the time evolution of r and χ obeys

$$\frac{d}{dt}\left\langle r^2\right\rangle = D_t(\varepsilon), \quad D_t(\varepsilon) = \frac{1}{6}\alpha^2 N^2 b^2 \varepsilon^{-2}, \tag{115}$$

$$\frac{d}{dt}\left\langle \tan^2(\chi)\right\rangle = D_s(\varepsilon), \quad D_s(\varepsilon) = \frac{1}{6}\alpha^4 N^2 b^2 \varepsilon^{-2}\omega_B^{-2}. \tag{116}$$

These equations immediately follow from (109) and above two results.

[47] Thereto from definitions (111) and (113), equations (115) and (116) ensure that the timescales of scattering, drift and energization of relativistic electrons are of the same order of magnitude. It should seem these events are not in any way connected. However, the above results clearly prove that the energy, pitch angle, and gyroradius changes are correlated due to the fact that a strange attractor dominates the dynamics of systems, such that the equations of motion, g^n and G^n, underlie FPK equations.

6. Application

[48] Enhanced convection electric fields associated with solar wind streams provide the principle mechanism for the intensification of ring current (10–100) keV flux, and also leads to the excitation of whistler mode waves [*Hastings and Garrett*, 1996]. The zone of most intense wave activity is spatially localized because of the decrease in resonant energy and wave guiding by strong density gradients associated with the plasmapause [*Wolf*, 1995]. Typical whistler wave amplitudes are in the range (10–100) pT, but occasionally the wave amplitude approaches 1 nT. The wave magnitudes of outer zone chorus emissions are usually sufficiently small that weak diffusion scattering results. Consequently, the electron heating should occur gradually over many drift orbits. A good introduction to physical aspects of the problem was given in the work of *Roth et al.* [1999]. According to *Roth et al.* [1999], the parameters of the problem are chosen to reflect a typical whistler wave interacting with electrons at locations outside the plasmapause, where the electron cyclotron frequency $\omega_B = 3 \cdot 10^4$ s^{-1}, electron plasma frequency $\omega_p = 6 \cdot 10^4$ s^{-1}, wave frequency $\omega = 1.5 \cdot 10^4$ s^{-1}, scale length of the zone of intense wave activity $L = 10^9$ cm, equatorial magnetic field of $1.5 \cdot 10^{-3}$ G, and magnitude of wave field of $(10^{-7} - 10^{-6})$ G. By (8) we find the phase velocity, $v_{ph} = 0.25$, group velocity, $v_{gr} = 2 \cdot v_{ph}(\omega_B - \omega)/\omega_B$, $v_{gr} \simeq 0.25$, resonance velocity $v_r = v_{ph}(\omega_B - \omega)/\omega$, $v_r \simeq 0.25$ and evaluate $\alpha = 2v_{ph}\omega_B/\omega$, $\alpha = 1.0$.

[49] We present our results starting with the nonrelativistic case. At first we evaluate the transit time of the wave packet through the domain of resonant interaction, $T = L/v_{gr}$, $T \simeq 1.3 \cdot 10^{-1}$, and consequently $N = [\omega T] = 2 \cdot 10^3$. Therefore by (65) and (83) the values of v_c and v_{th} at a given value of b, $b = 5 \cdot 10^{-5}$. Thus (65) yields $v_c \simeq 6 \cdot 10^{-4}$, the formula (83) can be represented as $v_{th} - v_r \simeq 10^{-15}/b^2$. This implies that the value of threshold speed is close to resonant value, therefore the electron remains near the exact resonance and may be accelerated. This leads to diffusion in energy and is accompanied by radial drift. Considering (103) and (93), we evaluate the rate of stochastic diffusion across the ambient magnetic field, $D_t \simeq 6 \cdot 10^9$ cm^2 s^{-1}, and the heating rate, $D_E = D\varepsilon mc^2$, $D_E \simeq 12.5$ keVs^{-1}. The efficiency of the process is highly significant. For instance, a classical Bohm-like coefficient, D^B, $D^B = \nu_c r^2$, ν_c ($\sim 10^{-14}$ s^{-1}) is the collision frequency, r ($\sim 10^5$ cm) is gyroradius at these locations is of order of $\sim 10^{-4}$ cm^2 s^{-1}. Now we estimate the threshold value of electron energy, $E_c = mv_{th}^2/2$, $E_c \simeq 15$ keV, whose value corresponds to the lower boundary of energy spectrum.

[50] We now turn to the relativistic case. To calculate the upper bound of the energy spectrum, we exploit expression (45) to find $E_b \simeq 8$ MeV at $b = 10^{-3}$. Note that E_b depends smoothly on b, therefore the energy spectrum is stable in a wide range of values of b. The restriction on b is conditioned only by requiring $\varepsilon_b^2 \gg \max\{\alpha^2, 1\}$. The latter yields

$$b > 4N^{-2} \quad \left(\sim 10^{-6}\right). \tag{117}$$

[51] As we discuss above, the heating rate of relativistic electrons by whistlers decreases with its energy steeply. This effect appears in the energy spectrum as follows. According to (105), the time to energize relativistic electrons is about 3 min, while the time for establishing the energy distribution in a range of nonrelativistic energies is approximately 10 s. Note that the characteristic time for the energy redistribution over the spectrum is small as compared with the typical period of intense wave activity because observations show that the power of whistler-mode waves is enhanced at substorm injection and decays over a period of hours or over a period of a few days during the storm recovery [*Baker et al.*, 1986; *Horne et al.*, 2003].

[52] From (113) at $\varepsilon_b = 16$ and $\alpha = 1$ follows that in this case whistler waves can diffuse electrons in a cone with the vertex angle, $\chi_v = 2\chi_b$, which is about 150°. The results (100) and (101) show that the wave packet effectively scatter nonrelativistic electrons in pitch angle leading to the establishment of a distribution peaked at $\chi_p \simeq 90°$. This must be associated with the so-called pancake distribution given by (54).

[53] These results demonstrate the possibility of electron heating over a wide energy range between $\geq$10 keV and a few MeV. This is in reasonable agreement with the experimental observations.

7. Summary

[54] A canonical Hamiltonian approach has been employed to deal with the whistler wave-electron interaction and the stochastic heating of high-energy electrons in magnetized plasmas. The time-like and space-like wave packet representations have been used in deriving the equations of motion for nonrelativistic and relativistic electrons, respectively. Phase dynamics of the system is shown to be realized on a stochastic attractor, all means on this attractor are stable and irrelevant to any initial conditions.

[55] The results afford a basis for conclusions. Solutions of the equations of motion have revealed the existence of an energetic threshold below which electron motion is regular. This threshold can be expressed in terms of the normalized magnitude of wave field, b, as $\varepsilon_c = const.b^{-4}$ as $b \to 0$, and $(\varepsilon_c - v_r^2/2) = const.b^{-2}$ as $b \to \infty$. The energy spectrum in a range of non-relativistic electron energies obeys the law $f(\varepsilon) = const./(\varepsilon - \varepsilon_c)$ which describes the order-chaos bifurcation transition at $\varepsilon = \varepsilon_c$. When the initial energy is greater than the energetic threshold, electrons move stochastically. Thus, nonlinear electron acceleration by a whistler wave packet is always a stochastic process. Chaotic motion gives rise to diffusion in energy and leads to establishing an energy spectrum, bounded above by ε_b. The upper value of energy spectrum has a weak dependence on $b(\propto b^{2/5})$.

[56] The obtained energetic spectra have been used for evaluating pitch angle distribution functions over different energies. Whistler mode waves can scatter nonrelativistic electrons with pitch angles of up to 90° and are responsible for the formation of the pancake distribution in the (v_z, v_t) phase space. Relativistic electrons are mainly scattered by these waves almost along the direction of an external magnetic field. Chaotic motion of electrons in a constant and homogeneous magnetic field is accompanied by radial drift. This effect is associated with stochastic heating of the particle. According to our analysis, for a spectrum of a whistler modes interacting nonlinearly with Earth's radiation belt electrons, substantial energization can occur in a

range of the energy from a few keV up to a few MeV on timescales in the order of a few minutes. High-energy electrons observed in Earth's radiation belts can be regarded as tails of the electron distributions in the magnetosphere, whose stability is ensured by the structural stability of the nonequilibrium wave-particle system. It is interesting to note similar features in electron motion are apparent in the Jovian radiation belts [*Khazanov et al.*, 2007]. It seems, evolution of both Jupiter's and Earth's radiation belts can be discussed as the manifestation of deterministic dynamics possessing a certain strange attractor. Thus stochastic processes occurring in the plasma can play an important role in the evolution of the energy spectra of radiation belts electrons.

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